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Alain Plagne, Anne de Roton

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ABSTRACT. In this paper, we are interested in a generalization of the notion of sum-free sets. We address a conjecture first made in the 90s by Chung and Goldwasser. Recently, after some computer checks, this conjecture was formulated again by Matolcsi and Ruzsa, who made a first significant step towards it. Here, we prove the full conjecture by giving an optimal upper bound for the Lebesgue measure of a 3-sum-free subset A of $[0, 1]$, that is, a set containing no solution to the equation $x+y = 3z$ where x, y and z are restricted to belong to A. We then address the inverse problem and characterize precisely, among all sets with that property, those attaining the maximal possible measure.

1. INTRODUCTION

Almost one century ago, on his way towards Fermat's Last Theorem, Schur [15] was led to study sets of integers not containing a triple of elements with one being the sum of the two others. This was a pioneering work in Ramsey theory but at the same time the study of such sets, now called *sum-free*, was seminal in combinatorial number theory. Since then, sum-free sets have been widely investigated (see for instance [17]) and generalized to further contexts.

One possible generalization is the study of sets of integers without solutions (x, y, z) to an equation of the form $ax + by = kz$ (with integral a, b and k) that Lucht [8] began in 1976. Ruzsa [12, 13] studied more general linear equations and introduced a new terminology by distinguishing between what he called *invariant* and *noninvariant* equations.

In the present paper, we shall only deal with k*-sum-free sets* (k is a positive integer). A subset of a given (additively written) semi-group, say, is said to be k*-sum-free* if it contains no triple (x, y, z) satisfying the equation $x + y = kz$. Invariant equations, which correspond here to the fact that the sum of the coefficients of the unknowns in the forbidden relation is equal to zero, lead to the existence of trivial solutions – as appears for instance in the case of 2-sum-free sets (since $x + x$ is equal to $2x$, whatever x is) – which have not to be considered and lead to special developments: 2-sum-free sets, which are also and in fact mainly known as sets without arithmetic progressions of length 3, are of great importance and their study is central in additive combinatorics. We simply mention [14] for the latest development on the subject which goes back at least to Roth [10]. Here, we restrict ourselves to the noninvariant cases, that is, k is

Both authors are supported by the ANR grant Cæsar, number ANR 12 - BS01 - 0011.

supposed different from 2. In this case, the problems which appear are of a different kind.

The very basic question to maximize the cardinality of a set of integers included in $\{1, 2, \ldots, n\}$ having no solution to the equation $x + y = z$ (sum-free sets) belongs to the folklore and is easily solved (see for instance [4] or [5]). One cannot select more than $\lceil n/2 \rceil$ integers with the required property, and this is optimal. Interestingly, for a general n, there are two kinds of extremal sum-free sets (see Theorem 1.1 of [4] for a precise statement): the *combinatorial* one, namely the upper-half, $\{\lceil (n+1)/2 \rceil, \ldots, n\}$ for which the impossibility to solve the equation follows from a size condition (the sum of two elements in this set is larger than n, thus outside the set); and the *arithmetic* one, in the present case the set of odd integers, for which a modular condition prevents from the existence of a solution. Not only in the case of sum-free sets of integers is this dichotomy emerging. In all these types of questions, when asked in a discrete setting, this typology is subject to appear.

For $k = 3$ (and $n \neq 4$), Chung and Goldwasser [4] proved Erdős' conjecture that $\lceil n/2 \rceil$ is the maximal size of a 3-sum-free set of positive integers less than n. They also prove, at least when $n \geq 23$ (see Theorem 1.3 in [4]), that the set of odd integers is the only example attaining this cardinality.

For $k \geq 4$, Chung and Goldwasser [3] discovered k-sum-free subsets of $\{1, 2, \ldots, n\}$ with a size asymptotic to

$$\sim \frac{k-2}{k^2-2}\left(k + \frac{8}{k(k^4-2k^2-4)}\right) n$$

as n tends to infinity. This was obtained thanks to an explicit construction of three intervals of integers. They additionally conjectured that this was the actual exact asymptotic maximal value. This conjecture was finally settled by Baltz, Hegarty, Knape, Larsson and Schoen in [1] (see also [6] for an alternative proof). These authors additionally proved an inverse theorem giving the structure of a k-sum-free sets of this size : such sets have to be close from the set composed of the three above-mentioned intervals.

In fact, Chung and Golwasser managed to predict the maximal size of a k-sum-free set of integers less than n by studying the continuous analog of the problem in [3]; in other words by introducing the study of k-sum-free subsets of real numbers selected from $[0, 1]$. Indeed, a k-sum-free subset of $[0, 1]$ leads, after a suitable dilation, to a k-sum-free set of integers (but it is important to notice, this set will be mandatorily – in the typology mentioned above – of a combinatorial nature).

We thus arrive to the question of determining the maximal Lebesgue measure – denoted thereafter μ – of a subset of $[0, 1]$ having no solution to the equation $x + y = kz$. The case $k = 1$ is easy and, as mentioned above, the cases $k \geq 4$ were solved in [3]. However, the case $k = 3$ was left open and remained the only one for which the optimal asymptotic density was unknown. Nonetheless, it was precisely investigated and the set

(composed again of three intervals)

$$\mathcal{A}_0 = \left(\frac{8}{177}, \frac{4}{59}\right) \cup \left(\frac{28}{177}, \frac{14}{59}\right) \cup \left(\frac{2}{3}, 1\right) \tag{1}$$

which does not contain a solution to the equation $x + y = 3z$, was identified in [3] as playing an important role in the question. Notice that its measure is equal to $77/177 = 0.4350\ldots$ In the sequel, we shall call $\mathcal{A}_1, \ldots, \mathcal{A}_7$ the seven sets defined as the union of $\mathcal{A}_0$ and three points, one end-point of each interval appearing in the definition of $\mathcal{A}_0$, except $(8/177, 14/59, 2/3)$. These seven sets are 3-sum-free. The quite precise following conjecture was then formulated in [3]:

Chung-Goldwasser Conjecture . *Let A be a measurable 3-sum-free subset of $[0,1]$. Then*

$$\mu(A) \leq \frac{77}{177}.$$

Moreover, if $\mu(A) = 77/177$ and if A is maximal with respect to inclusion among the 3-sum-free subsets of $[0,1]$, then $A \in \{\mathcal{A}_1, \ldots, \mathcal{A}_7\}$.

Recently, Matolcsi and Ruzsa [9] led several computer-aided checks supporting this conjecture. Mainly, they made the first breakthrough towards the first part of this conjecture by showing the following theorem.

Matolcsi-Ruzsa Theorem . *Let A be a mesurable 3-sum-free subset of $[0,1]$. Then its measure satisfies*

$$\mu(A) \leq \frac{28}{57} = 0.49122\ldots$$

This result, the first one to prove a strictly less than 0.5 upper bound for 3-sum-free subsets of $[0,1]$, is very noticeable because it shows in particular that in the case of 3-sum-free sets, contrary to what happens in the other cases, the maximal size of such a subset in $[0,1]$ is not the analog of that of a k-sum-free subset of integers (let us recall that such a set has a density $1/2$). This illustrates indeed the fact that the only known 3-sum-free set of integers of maximal size is the set of odd numbers, a set of an arithmetic nature which does not possess a 'direct' continuous analog (contrarily to sets of combinatorial nature). Well, one could try to fatten up the set of odd numbers in $\{1, 2, \cdots, n\}$ and normalize it to obtain a 3-sum free subset of $[0,1]$ as explained in [3] but the measure of this set would be roughly $1/3$. This is an important observation: while such a phenomenon may occur when comparing sets with no solution (to a given linear equation) in $\{1, 2, \cdots, n\}$ and in $[0,1]$, a recent theorem of Candela and Sisask (see Theorem 1.3 in [2]) shows that, in the analogous case of cyclic groups of prime order versus the torus, the discrete model always converges towards the continuous one. Notice that a good reason for this to happen, in this discrete case, is that even sets of an arithmetic nature can be transformed without loss of generality into sets of a combinatorial nature with the same density: multiplying an arithmetic progression by the inverse of its difference transforms it into an interval. This does not happen in the case of present study and makes the behaviour of maximal sets more difficult to handle.

In this paper, we first establish the optimal (in view of example (1)) upper bound for the measure of 3-sum-free sets of $[0, 1]$.

Theorem 1. *Let A be a measurable 3-sum-free subset of $[0, 1]$. Then*

$$\mu(A) \leq \frac{77}{177}.$$

We then describe the 3-sum-free subsets of $[0, 1]$ of maximal measure, which in particular solves the inverse associated problem.

Theorem 2. *Let A be a measurable 3-sum-free subset of $[0, 1]$ satisfying $\mu(A) = 77/177$, then there is an $i \in \{1, \dots, 7\}$ such that $A \subset \mathcal{A}_i$.*

The full Chung-Goldwasser conjecture is thus proved.

2. Notations and prerequisites

In what follows, we denote respectively by $\mu(X)$ and $\mathrm{diam(X)} = \sup \mathrm{X} - \inf \mathrm{X}$, the Lebesgue measure and the diameter of a set X of real numbers. We shall denote by $A + B$ the Minkowski sumset of two subsets A and B of $\mathbb{R}$, and by $\alpha \cdot A$ the α-dilate of A, that is $\{\alpha x \text{ for } x \in A\}$. Notice in particular that $2 \cdot A$ is included in, but in general different from, $A + A$.

While the behaviour of μ with respect to dilation is clear since one has

$$\mu(\alpha \cdot A) = \alpha \mu(A), \tag{2}$$

it is more complicated for the case of Minkowski addition. The basic estimate for the measure of the sum of two measurable bounded subsets A and B of $\mathbb{R}$ is a standard Brunn-Minkowski type [7] lower bound, namely

$$\mu_*(A + B) \geq \mu(A) + \mu(B), \tag{3}$$

where μ_* denotes the inner measure (the use of this tool is made necessary by the fact that $A + B$ is not necessarily measurable as shown by Sierpiński [16]).

Beyond this, the best known result is due to Ruzsa.

Lemma 1. *(Ruzsa [11]) Let A and B be two bounded measurable subsets of $\mathbb{R}$ such that $\mu(A) \leq \mu(B)$, then*

$$\mu_*(A + B) \geq \min(2\mu(A) + \mu(B), \mu(A) + \mathrm{diam}(B)). \tag{4}$$

In particular, one has

$$\mu_*(A + A) \geq \min(3\mu(A), \mu(A) + \mathrm{diam}(A)). \tag{5}$$

We underline the fact that up to this lemma, the result presented in this paper is self-contained.

We now state two more specific lemmas, due to Matolcsi and Ruzsa [9], that we shall need in the present study. These intermediary results are not presented as lemmas in [9], therefore, to ease the reading of the present paper, we include their respective (condensed) proofs here. Before entering this, we note that the assumption that there is no solution to $x + y = 3z$ with $x, y, z \in A$ can be rewritten set-theoretically in the form

$$(A + A) \cap (3 \cdot A) = \emptyset \quad \text{or, equivalently,} \quad \left(\frac{1}{3} \cdot (A + A)\right) \cap A = \emptyset. \tag{6}$$

Lemma 2. *(Matolcsi-Ruzsa* [9]*) Let A be a measurable bounded 3-sum-free subset of $\mathbb{R}^+$. One has*

$$\mu(A) \leq \frac{2 \sup A - \inf A}{4}.$$

Proof. Since, by definition, $1/3 \cdot (A + A)$ and A are intersection-free and both included in the interval $[2 \inf A/3, \sup A]$, we obtain

$$\begin{aligned} \sup A - \frac{2}{3} \inf A &\geq \mu_* \left(\frac{1}{3} \cdot (A + A)\right) + \mu(A) \\ &\geq \min \left(\mu(A), \frac{1}{3}(\mu(A) + \sup A - \inf A)\right) + \mu(A), \end{aligned}$$

in view of (2) and (5). If $2\mu(A) \geq \operatorname{diam}(A)$, then we obtain

$$\sup A - \frac{2}{3} \inf A \geq \frac{4}{3}\mu(A) + \frac{1}{3}(\sup A - \inf A)$$

which gives the result. In the other case, we have

$$\mu(A) \leq \frac{1}{2}\operatorname{diam}(A) = \frac{1}{2}(\sup A - \inf A) \leq \frac{2 \sup A - \inf A}{4},$$

since $A \subset \mathbb{R}^+$. □

Here is the second lemma useful to our purpose.

Lemma 3. *(Matolcsi-Ruzsa* [9]*) Let A be a measurable 3-sum-free subset of $[0, 1]$ such that $\sup A = 1$, then*

$$\mu(A) \leq \frac{1}{3} + \frac{1}{2}\mu \left(A \cap \left[\frac{2}{3}, 1\right]\right).$$

Proof. We define

$$a = \inf A, \quad A_1 = A \cap \left[\frac{2}{3}, 1\right] \quad \text{and} \quad \varepsilon = \frac{1}{3} - \mu(A_1)$$

and refine the argument used in the proof of Lemma 2 using that the three sets $1/3 \cdot (A+A)$, A and $(2/3,1] \setminus A_1$ are disjoint subsets of $[2a/3,1]$. This gives

$$1-\frac{2a}{3} \geq \frac{1}{3}\mu_*(A+A)+\mu(A)+\varepsilon.$$

Now, (5) gives

$$\mu_*(A+A) \geq \mu(A)+\min(2\mu(A),1-a).$$

In the case where $2\mu(A) > 1-a$, we get

$$1-\frac{2a}{3} \geq \frac{1-a}{3}+\frac{4}{3}\mu(A)+\varepsilon,$$

thus

$$\mu(A) \leq \frac{1}{2}-\frac{3}{4}\varepsilon-\frac{a}{4},$$

whereas if $2\mu(A) \leq 1-a$, we get

$$1-\frac{2a}{3} \geq 2\mu(A)+\varepsilon,$$

from which it follows that

$$\mu(A) \leq \frac{1-\varepsilon}{2}-\frac{a}{3}.$$

In both cases, we have

$$\mu(A) \leq \frac{1-\varepsilon}{2}=\frac{1}{3}+\frac{1}{2}\mu(A_1),$$

that is, the result. □

3. Two central lemmas

The proof of the main theorem (Theorem 1) relies essentially on the following technical lemma.

Lemma 4. *Let A be a measurable 3-sum-free subset of $[0,1]$ such that $\sup A = 1$. Let*

$$a=\inf A \quad \textit{and} \quad A_1 = A \cap \left[\frac{2}{3},1\right]$$

and define

$$\varepsilon_1 = \inf A_1 - \frac{2}{3} \qquad \textit{and} \qquad \varepsilon_2 = \left(\frac{1}{3}-\varepsilon_1\right)-\mu(A_1).$$

If $\varepsilon_1+2\varepsilon_2 \leq 1/3$, then one has

$$\mu\left(A \cap \left[\frac{2}{9}+\frac{a}{3},1\right]\right) \leq \begin{cases} \frac{1}{3}-\frac{1}{6}\varepsilon_1 & \textit{if } \varepsilon_1 \leq \frac{2}{3}a, \\ \frac{1}{3}-\frac{1}{24}\left(\varepsilon_1-\frac{2}{3}a\right) & \textit{if } \varepsilon_1 > \frac{2}{3}a. \end{cases}$$

Proof. We define the following three sets $A_{2/3}$, $A_{4/9}$ and $A_{1/3}$:

$$A_{2/3} = A \cap \left[\frac{4}{9} + \frac{2}{3}\varepsilon_1, \frac{2}{3}\right], \quad A_{4/9} = A \cap \left[\frac{1}{3} + \frac{1}{2}\varepsilon_1, \frac{4}{9} + \frac{1}{6}\varepsilon_1\right],$$

and

$$A_{1/3} = A \cap \left[\frac{2}{9} + \frac{1}{3}(a+\varepsilon_1), \frac{1}{3} + \frac{1}{3}a\right].$$

By (5), one gets

$$\begin{aligned} \mu_*\left(\frac{1}{3}\cdot(A_1+A_1)\right) &\geq \min\left(\mu(A_1), \frac{1}{3}(\mu(A_1) + \mathrm{diam}(A_1))\right) \\ &= \min\left(\frac{1}{3} - \varepsilon_1 - \varepsilon_2, \frac{1}{3}\left(\frac{2}{3} - 2\varepsilon_1 - \varepsilon_2\right)\right) \\ &= \frac{2}{9} - \frac{2}{3}\varepsilon_1 - \frac{1}{3}\varepsilon_2 \qquad (7) \end{aligned}$$

using the assumption that $\varepsilon_1 + 2\varepsilon_2 \leq 1/3$.

Since, by (6), the two sets $1/3\cdot(A_1+A_1)$ and $A_{2/3}$ are disjoint subsets of the interval $[4/9 + 2\varepsilon_1/3, 2/3]$, one obtains, using (7),

$$\mu(A_{2/3}) \leq \mu\left(\left[\frac{4}{9} + \frac{2}{3}\varepsilon_1, \frac{2}{3}\right]\right) - \mu_*\left(\frac{1}{3}\cdot(A_1+A_1)\right) \leq \frac{\varepsilon_2}{3}. \qquad (8)$$

We now prove that

$$\mu(A_{4/9}) \leq \frac{1}{3}\varepsilon_2. \qquad (9)$$

If $A_{4/9}$ has measure zero, there is nothing to prove. Thus we denote by c_1 the infimum of $A_{4/9}$ and by c_2 its supremum, and assume they are distinct. We choose a decreasing sequence $(c_1(n))_{n\geq 0}$ in $A_{4/9}$ tending to c_1 when n tends to infinity and $< c_2$ (if c_1 is in $A_{4/9}$, $c_1(n) = c_1$ will do). One has in view of $1/3 + \varepsilon_1/2 \leq c_1 \leq c_1(n) < c_2 \leq 4/9 + \varepsilon_1/6$,

$$\frac{1}{3}\left(c_1(n) + \frac{2}{3} + \varepsilon_1\right) \leq c_1(n) \leq c_2 \leq \frac{4}{9} + \frac{\varepsilon_1}{6} \leq \frac{1+c_1}{3} \leq \frac{1+c_1(n)}{3},$$

therefore

$$\begin{aligned} A_{4/9} &= A \cap [c_1, c_2] \\ &= (A \cap [c_1, c_1(n)]) \cup (A \cap [c_1(n), c_2]) \\ &\subset (A \cap [c_1, c_1(n)]) \cup \left(A \cap \left[\frac{1}{3}\left(c_1(n) + \frac{2}{3} + \varepsilon_1\right), \frac{1}{3}(c_1(n)+1)\right]\right) \\ &\subset (A \cap [c_1, c_1(n)]) \cup \left(A \cap \frac{1}{3}\cdot\left(c_1(n) + \left[\frac{2}{3} + \varepsilon_1, 1\right]\right)\right). \end{aligned}$$

Since $c_1(n) \in A$, assumption (6) implies that $\frac{1}{3}(c_1(n)+A_1) \cap A_{4/9} = \emptyset$ and therefore

$$A_{4/9} \subset (A \cap [c_1, c_1(n)]) \cup \frac{1}{3} \cdot \left(c_1(n) + \left(\left[\frac{2}{3} + \varepsilon_1, 1 \right] \setminus A_1 \right) \right), \tag{10}$$

which in turn gives

$$\begin{aligned} \mu(A_{4/9}) &\leq \mu([c_1, c_1(n)]) + \frac{1}{3}\mu\left(\left[\frac{2}{3} + \varepsilon_1, 1 \right] \setminus A_1 \right) \\ &= (c_1(n) - c_1) + \frac{1}{3}\left(\frac{1}{3} - \varepsilon_1 - \mu(A_1) \right) \\ &= (c_1(n) - c_1) + \frac{1}{3}\varepsilon_2. \end{aligned}$$

Letting n tend to infinity in this inequality finishes the proof of (9).

In the same fashion, if $a \in A$, one obtains

$$A_{1/3} = A \cap \frac{1}{3} \cdot \left(a + \left[\frac{2}{3} + \varepsilon_1, 1 \right] \right) = A \cap \frac{1}{3} \cdot \left(a + \left(\left[\frac{2}{3} + \varepsilon_1, 1 \right] \setminus A_1 \right) \right)$$

from which it follows

$$\mu(A_{1/3}) \leq \frac{1}{3}\varepsilon_2 \tag{11}$$

and this remains true even if $a \notin A$ by considering a sequence $(a(n))_{n \geq 0}$ of elements of A tending to a when n goes to infinity arguing similarly as in the proof of (9).

We now study separately the two inequalities in the statement of the Lemma.

First inequality.

Suppose first that $\varepsilon_1 \leq 2a/3$, which implies

$$\sup \left[\frac{2}{9} + \frac{1}{3}(a + \varepsilon_1), \frac{1}{3} + \frac{1}{3}a \right] \geq \inf \left[\frac{1}{3} + \frac{1}{2}\varepsilon_1, \frac{4}{9} + \frac{1}{6}\varepsilon_1 \right],$$

or, in other words, that $A_{4/9}$ and $A_{1/3}$ overlap. One then deduces from (8), (9) and (11) that

$$\begin{aligned} \mu\left(A \cap \left[\frac{2}{9} + \frac{a}{3}, 1 \right] \right) &\leq \mu(A_1) + \mu(A_{2/3}) + \frac{1}{2}\varepsilon_1 + \mu(A_{4/9}) + \mu(A_{1/3}) + \frac{1}{3}\varepsilon_1 \\ &\leq \left(\frac{1}{3} - \varepsilon_1 - \varepsilon_2 \right) + \varepsilon_2 + \frac{5}{6}\varepsilon_1 = \frac{1}{3} - \frac{1}{6}\varepsilon_1. \end{aligned}$$

And the inequality of Lemma 4 follows in this first case.

Second inequality.

Until the end of this proof, we assume that $\varepsilon_1 > 2a/3$.

We shall need the sets $B_{1/3}$ and $C_{1/3}$ defined in the following way :

$$B_{1/3}=A\cap\left[\frac{1}{3}+\frac{1}{3}a,\frac{1}{3}+\frac{1}{2}\varepsilon_1\right],\quad C_{1/3}=A\cap\left[\frac{2}{9}+\frac{2}{9}a,\frac{2}{9}+\frac{1}{3}\varepsilon_1\right].$$

The assumption on the relative sizes of ε_1 and a shows that

$$\frac{1}{3}\cdot\left(\frac{2}{3}+\varepsilon_1+B_{1/3}\right)\subset\left[\frac{1}{3}+\frac{a}{9}+\frac{\varepsilon_1}{3},\frac{1}{3}+\frac{\varepsilon_1}{2}\right]\subset\left[\frac{1}{3}+\frac{a}{3},\frac{1}{3}+\frac{\varepsilon_1}{2}\right].$$

If $2/3+\varepsilon_1\in A$, (6) shows that the set on the left is intersection-free with $B_{1/3}$ and one thus gets

$$\mu(B_{1/3})\le\mu\left(\left[\frac{1}{3}+\frac{a}{3},\frac{1}{3}+\frac{\varepsilon_1}{2}\right]\right)-\frac{1}{3}\mu(B_{1/3})=\frac{\varepsilon_1}{2}-\frac{a}{3}-\frac{\mu(B_{1/3})}{3},\tag{12}$$

consequently

$$\mu(B_{1/3})\le\frac{3}{4}\left(\frac{\varepsilon_1}{2}-\frac{a}{3}\right).\tag{13}$$

Once again, the same type of arguments as the ones used to prove (9) shows that this remains true even if $2/3+\varepsilon_1\notin A$.

Now the inclusion

$$\frac{1}{3}\cdot(B_{1/3}+B_{1/3})\subset\left[\frac{2}{9}+\frac{2a}{9},\frac{2}{9}+\frac{\varepsilon_1}{3}\right]$$

and (6) show that the set on the left-hand side of this inclusion and $C_{1/3}$ are disjoint and both included in the set on the right-hand side, that is, we obtain

$$\mu_*\left(\frac{1}{3}\cdot(B_{1/3}+B_{1/3})\right)+\mu(C_{1/3})\le\frac{\varepsilon_1}{3}-\frac{2a}{9}.\tag{14}$$

By (3), this yields

$$\frac{2}{3}\mu\left(B_{1/3}\right)+\mu(C_{1/3})\le\frac{\varepsilon_1}{3}-\frac{2a}{9}.$$

Then, using this and (13), we derive

$$\begin{aligned}\mu(B_{1/3})+\mu(C_{1/3})&=\left(\frac{2}{3}\mu\left(B_{1/3}\right)+\mu(C_{1/3})\right)+\frac{1}{3}\mu\left(B_{1/3}\right)\\&\le\frac{\varepsilon_1}{3}-\frac{2a}{9}+\frac{1}{4}\left(\frac{\varepsilon_1}{2}-\frac{a}{3}\right)\\&=\frac{11}{24}\varepsilon_1-\frac{11}{36}a.\end{aligned}$$

We finally deduce from (8), (9), (11) and the preceding inequality, that

$$\begin{aligned}\mu\left(A\cap\left[\frac{2}{9}+\frac{a}{3},1\right]\right) &\leq \mu(A_1)+\mu(A_{2/3})+\frac{\varepsilon_1}{2}+\mu(A_{4/9})+\mu(B_{1/3})+\mu(A_{1/3})\\ &\qquad +\frac{a}{3}+\mu(C_{1/3})\\ &\leq \left(\frac{1}{3}-\varepsilon_1-\varepsilon_2\right)+\varepsilon_2+\frac{1}{2}\varepsilon_1+\frac{1}{3}a+\frac{11}{24}\varepsilon_1-\frac{11}{36}a\\ &= \frac{1}{3}-\frac{1}{24}\varepsilon_1+\frac{1}{36}a\\ &= \frac{1}{3}-\frac{1}{24}\left(\varepsilon_1-\frac{2}{3}a\right).\end{aligned}$$

Hence the announced inequality. □

The second central lemma, needed for the proof of Theorem 2, deals with attaining the bound $1/3$ in Lemma 4. Here it is.

Lemma 5. *Let A be a measurable 3-sum-free subset of $[0,1]$ such that $\sup A=1$. We define*

$$a=\inf A,\quad A_1=A\cap\left[\frac{2}{3},1\right],\quad \varepsilon_1=\inf A_1-\frac{2}{3}\quad \textit{and}\quad \varepsilon_2=\left(\frac{1}{3}-\varepsilon_1\right)-\mu(A_1).$$

We assume $\varepsilon_1+2\varepsilon_2\leq 1/3$ and $a>0$. Then, $\mu\left(A\cap[2/9+a/3,1]\right)=1/3$ implies

$$\varepsilon_1=\varepsilon_2=0.$$

Proof. In this proof we will use freely the notation introduced in the preceding lemma.

We first apply Lemma 4 to A. The precise inequality obtained there implies that we cannot have $\varepsilon_1>2a/3$, since we would get $\mu(A\cap[2/9+a/3,1])<1/3$. Thus $\mu(A\cap[2/9+a/3,1])=1/3$ implies $\varepsilon_1\leq 2a/3$ and then

$$\varepsilon_1=0$$

in view of the precise formula in this case.

We now turn to the core of this proof and show that

$$\varepsilon_2=0 \tag{15}$$

and assume for a contradiction that $\varepsilon_2>0$.

The definition of the sets introduced in Lemma 4 gives

$$\mu\left(A\cap\left[\frac{2}{9}+\frac{a}{3},1\right]\right)=\mu(A_1)+\mu(A_{2/3})+\mu(A_{4/9})+\mu(A_{1/3})-\mu(A_{1/3}\cap 4_{4/9}).$$

Recall that by definition $\mu(A_1)=1/3-\varepsilon_2$ and, in view of (8), (9) and (11),

$$\mu(A_{1/3}),\ \mu(A_{4/9}),\ \mu(A_{2/3})\leq\frac{\varepsilon_2}{3}.$$

This, with $\mu(A_{1/3} \cap 4_{4/9}) \geq 0$, shows that $\mu\left(A \cap [2/9 + a/3, 1]\right) = 1/3$ can hold only in the case

$$\mu(A_{1/3}) = \mu(A_{4/9}) = \mu(A_{2/3}) = \frac{\varepsilon_2}{3}, \quad \text{and} \quad \mu\left(\left[\frac{1}{3}, \frac{1}{3} + \frac{a}{3}\right] \cap A\right) = 0, \tag{16}$$

this last equality being tantamount to saying that the intersection of $A_{1/3}$ and $A_{4/9}$ has measure zero.

The function $f : \big(x \mapsto \mu([1/3, x] \cap A_{4/9})\big)$ is a non-decreasing non-negative continuous function on $[1/3, 4/9]$ such that f is identically 0 on $[1/3, (1+a)/3]$ and $f(4/9) = \varepsilon_2/3 > 0$. We define $\tilde{c_1}$ as the following infimum

$$\tilde{c_1} = \inf\{x \in [1/3, 4/9], f(x) > 0\}.$$

We have

$$\tilde{c_1} \geq \frac{1+a}{3}. \tag{17}$$

Furthermore, $\mu([1/3, x] \cap A_{4/9}) = 0$ for any $x \in [1/3, \tilde{c_1}]$, whereas $\mu\left(A_{4/9} \cap [\tilde{c_1}, \tilde{c_1} + \eta]\right) > 0$ for any $\eta > 0$.

We choose a real number η such that $0 < \eta < \min(a, \varepsilon_2)/3$. Let v be any element of $[\tilde{c_1}, \tilde{c_1} + \eta] \cap A_{4/9}$. We have, using (17),

$$\frac{1}{3} < v < \tilde{c_1} + \eta < \tilde{c_1} + \frac{a}{3} \leq \tilde{c_1} + \left(\tilde{c_1} - \frac{1}{3}\right) < \tilde{c_1} + 2\left(\tilde{c_1} - \frac{1}{3}\right) = 3\tilde{c_1} - \frac{2}{3}$$

from which it follows that

$$\frac{1}{3}\left(v + \frac{2}{3}\right) \leq \tilde{c_1} \leq c_2 \leq \frac{4}{9} \leq \frac{1}{3}(v+1),$$

on recalling that $c_2 = \sup A_{4/9}$. Going back to the proof that $\mu(A_{4/9}) = \varepsilon_2/3$ in Lemma 4, the preceding inequalities allow us to obtain

$$A_{4/9} \subset (A \cap [c_1, \tilde{c_1}]) \cup (A \cap [\tilde{c_1}, c_2]) \subset (A \cap [c_1, \tilde{c_1}]) \cup \frac{1}{3} \cdot \left(A \cap \left(v + \left[\frac{2}{3}, 1\right]\right)\right).$$

As previously, assumption (6) yields

$$A_{4/9} \subset (A \cap [c_1, \tilde{c_1}]) \cup \frac{1}{3} \cdot \left(v + \left(\left[\frac{2}{3}, 1\right] \setminus A_1\right)\right).$$

But the sets $A_{4/9}$ and $1/3 \cdot (v + ([2/3, 1] \setminus A_1))$ have the same measure $\varepsilon_2/3$ while $A \cap [c_1, \tilde{c_1}]$ has measure zero. We therefore deduce that, for any v in $[\tilde{c_1}, \tilde{c_1} + \eta] \cap A_{4/9}$,

$$A_{4/9} = \frac{1}{3} \cdot \left(v + \left(\left[\frac{2}{3}, 1\right] \setminus A_1\right)\right) \tag{18}$$

up to a set of measure zero. Choosing $u \neq u'$ in $[\tilde{c_1}, \tilde{c_1}+\eta] \cap A_{4/9}$ (such u and u' do exist since $\mu\left([\tilde{c_1}, \tilde{c_1}+\eta] \cap A\right) \neq 0$), and applying (18) consecutively to $v = u$ and $v = u'$, we get, up to sets of measure zero,

$$u + \left(\left[\frac{2}{3}, 1\right] \setminus A_1\right) = u' + \left(\left[\frac{2}{3}, 1\right] \setminus A_1\right).$$

This implies that

$$\varepsilon_2 = 3\mu\left(\left[\frac{2}{3}, 1\right] \setminus A_1\right) = 0,$$

a contradiction. Assertion (15) is therefore proved. □

4. Proof of Theorem 1

Let us begin with a simple consequence of Lemma 4.

Lemma 6. *Let A be a measurable 3-sum-free subset of $[0,1]$ such that $\sup A = 1$ and $\mu(A) \geq 5/12$. Then,*

$$\mu\left(A \cap \left[\frac{\inf A}{3} + \frac{2}{9}, 1\right]\right) \leq \frac{1}{3}.$$

Proof. Define ε_1 and ε_2 as in the statement of Lemma 4. The assumptions and Lemma 3 show that

$$\frac{5}{12} \leq \mu(A) \leq \frac{1}{3} + \frac{1}{2}\mu\left(A \cap \left[\frac{2}{3}, 1\right]\right) = \frac{1}{2}(1 - \varepsilon_1 - \varepsilon_2),$$

or, equivalently, $\varepsilon_1 + \varepsilon_2 \leq 1/6$, which implies $\varepsilon_1 + 2\varepsilon_2 \leq 2(\varepsilon_1 + \varepsilon_2) \leq 1/3$ and makes it possible to apply Lemma 4, which in turn concludes the proof. □

We can now prove Theorem 1, the main result of this paper.

Proof of Theorem 1. We may, without loss of generality, assume that $\sup(A) = 1$, since otherwise, we consider $(1/\sup(A)) \cdot A$. Since $77/177 > 5/12$, we may also assume that $\mu(A) \geq 5/12$, otherwise there is nothing to prove. Therefore, applying Lemma 6, we get

$$\mu(A) \leq \frac{1}{3} + \mu(R) \quad \text{where} \quad R = A \cap \left[a, \frac{2}{9} + \frac{1}{3}a\right] \quad \text{and} \quad a = \inf A. \tag{19}$$

Notice that

$$\mu(R) \leq \mu\left(\left[a, \frac{2}{9} + \frac{1}{3}a\right]\right) = \frac{2}{9} - \frac{2a}{3}. \tag{20}$$

Since R is non-empty (its measure is at least $1/12$, by assumption), we define

$$r = \sup R, \quad R' = \frac{1}{r} \cdot R \quad \text{and} \quad R'_1 = R' \cap \left[\frac{2}{3}, 1\right] = \frac{1}{r} \cdot \left(R \cap \left[\frac{2}{3}r, r\right]\right)$$

and put

$$\eta_1 = \inf R'_1 - \frac{2}{3}, \quad \eta_2 = \frac{1}{3} - \eta_1 - \mu(R'_1).$$

We distinguish two cases.

Case 1: $\eta_1+2\eta_2\le 1/3$.

We apply Lemma 4 to the set R' and get

$$\mu\left(R'\right)\le\frac{1}{3}+\mu\left(R'\cap\left[\frac{a}{r},\frac{2}{9}+\frac{a}{3r}\right]\right).$$

This implies

$$\mu\left(R\right)\le\frac{r}{3}+\mu\left(R\cap\left[a,\frac{2r}{9}+\frac{a}{3}\right]\right)=\frac{r}{3}+\mu\left(R_0\right),\tag{21}$$

if we denote

$$R_0=R\cap\left[a,\frac{2r}{9}+\frac{a}{3}\right].$$

If $\mu(R_0)=0$, then by (20), (21) and the inequality $r\le 2/9+a/3$, we obtain

$$\mu(R)\le\min\left(\frac{r}{3},\frac{2}{9}-\frac{2a}{3}\right)\le\min\left(\frac{2}{27}+\frac{a}{9},\frac{2}{9}-\frac{2a}{3}\right)\le\frac{2}{21}.$$

This can be easily checked by noticing that the maximum value is attained for $a=4/21$. Thus, in this case we must have

$$\mu(A)\le\frac{1}{3}+\frac{2}{21}=\frac{3}{7}<\frac{77}{177}$$

and we are done.

From now on, we therefore assume that $\mu(R_0)>0$, in particular that R_0 is a non empty set and we define $b=\sup R_0$. Applying Lemma 2 to R_0 together with the obvious inequality $\mu(R_0)\le b-a$ yields

$$\begin{aligned}\mu(R_0)&\le\min\left(\frac{2b-a}{4},b-a\right)\\&\le\min\left(\frac{r}{9}-\frac{1}{12}a,\frac{2r}{9}-\frac{2}{3}a\right)\end{aligned}$$

since $b\le 2r/9+a/3$. Therefore, by (21), we have

$$\mu(R)\le\min\left(\frac{4r}{9}-\frac{1}{12}a,\frac{5r}{9}-\frac{2}{3}a\right).$$

Using (19) and $r \leq 2/9 + a/3$, we get

$$\begin{aligned}\mu(A) &\leq \frac{1}{3} + \min\left(\frac{4}{9}\left(\frac{2}{9}+\frac{a}{3}\right) - \frac{1}{12}a, \frac{5}{9}\left(\frac{2}{9}+\frac{1}{3}a\right) - \frac{2}{3}a\right) \\ &\leq \frac{1}{3} + \min\left(\frac{8}{81}+\frac{7}{108}a, \frac{10}{81} - \frac{13}{27}a\right) \\ &\leq \frac{77}{177}.\end{aligned}$$

Moreover, the upper bound is tight and taken uniquely by the value $a = 8/177$. This piece of information will be used later on in the proof of Theorem 2.

<u>Case 2</u> : Assume now that $\eta_1 + 2\eta_2 > 1/3$.

In particular, $\eta_1 + \eta_2 > 1/6$. This together with Lemma 3 give $\mu(R') \leq 5/12$, thus

$$\mu(R) \leq \frac{5r}{12}. \tag{22}$$

We now prove that

$$\mu(R) \leq \max\left(\frac{1}{2}(r-a), \frac{2r-a}{6}\right). \tag{23}$$

Indeed, if $\mu(R) > (r-a)/2 = \mathrm{diam}(R)/2$, then (5) implies

$$\mu_*(R+R) \geq \mu(R) + \mathrm{diam}(R) = \mu(R) + (r-a).$$

Since $1/3 \cdot (R+R) \subset [2a/3, 2r/3]$ and $(1/3 \cdot (R+R)) \cap R = \emptyset$, we get

$$\mu\left(R \cap \left[a, \frac{2}{3}r\right]\right) = \mu\left(R \cap \left[\frac{2}{3}a, \frac{2}{3}r\right]\right) \leq \frac{2}{3}(r-a) - \mu_*\left(\frac{1}{3}\cdot(R+R)\right) \leq \frac{1}{3}(r-a) - \frac{1}{3}\mu(R).$$

It follows that

$$\begin{aligned}\mu(R) &= \mu\left(R \cap \left[a, \frac{2}{3}r\right]\right) + \mu\left(R \cap \left[\frac{2}{3}r, r\right]\right) \\ &\leq \frac{1}{3}(r-a) - \frac{1}{3}\mu(R) + \frac{r}{3} - (\eta_1+\eta_2)r \\ &\leq \frac{1}{3}(r-a) - \frac{1}{6}(r-a) + \frac{r}{6} \\ &= \frac{2r-a}{6}\end{aligned}$$

and assertion (23) is proved.

Synthetizing (22), (19) and (23), we finally obtain

$$\mu(A) \leq \frac{1}{3} + \min\left(\max\left(\frac{1}{2}(r-a), \frac{1}{3}r - \frac{1}{6}a\right), \frac{5}{12}r\right).$$

Taking into account $r \leq 2/9 + a/3$, we get

$$\mu(A) \leq \frac{1}{3} + \min\left(\max\left(\frac{1}{9} - \frac{1}{3}a, \frac{2}{27} - \frac{1}{18}a\right), \frac{5}{54} + \frac{5}{36}a\right).$$

If $a < 2/15$, this yields

$$\mu(A) \leq \frac{1}{3} + \min\left(\frac{1}{9} - \frac{1}{3}a, \frac{5}{54} + \frac{5}{36}a\right) \leq \frac{22}{51}.$$

This can be checked by noticing that the maximal value of the minimum is attained for $a = 2/51$. If $a \geq 2/15$, we have

$$\mu(A) \leq \frac{1}{3} + \min\left(\frac{2}{27} - \frac{1}{18}a, \frac{5}{54} + \frac{5}{36}a\right) = \frac{1}{3} + \left(\frac{2}{27} - \frac{1}{18}a\right) \leq \frac{1}{3} + \frac{2}{27} = \frac{11}{27}.$$

Since both $22/51$ and $11/27$ are $< 77/177$, we obtain, in this second case, that $\mu(A) < 77/177$.

This concludes the proof of Theorem 1.

□

5. The inverse result: proof of Theorem 2

This section is devoted to the proof of the structural characterization of 3-sum-free sets with maximal measure. We start with a lemma which contains the core of the structural result.

Lemma 7. *Let A be a measurable 3-sum-free subset of $[0, 1]$ satisfying $\mu(A) = 77/177$. Then $\mu(A\Delta\mathcal{A}_0) = 0$ where $A\Delta\mathcal{A}_0$ stands for the symmetric difference between A and $\mathcal{A}_0$ as defined in formula* (1)*.*

Proof. Let us assume that we have a set $A \subset [0, 1]$ with no solution to the equation $x + y = 3z$ such that $\mu(A) = 77/177$. We can assume that $\sup(A) = 1$, otherwise $(1/\sup A) \cdot A$ would contradict Theorem 1.

For the sake of clarity, we recall the notation we shall use in this proof, namely

$$a = \inf A, \quad A_1 = A \cap \left[\frac{2}{3}, 1\right], \quad \varepsilon_1 = \inf A_1 - \frac{2}{3}, \quad \varepsilon_2 = \frac{1}{3} - \varepsilon_1 - \mu(A_1),$$

$$R = A \cap \left[a, \frac{2}{9} + \frac{1}{3}a\right], \quad r = \sup R, \quad R_1' = \left(\frac{1}{r} \cdot R\right) \cap \left[\frac{2}{3}, 1\right],$$

and

$$R_0 = R \cap \left[a, \frac{2}{9}r + \frac{a}{3}\right], \quad b = \sup R_0.$$

If we examine the proof of Theorem 1, we notice first that we must have $\mu(A) = 1/3 + \mu(R)$ that is,

$$\mu\left(A \cap \left[\frac{2}{9} + \frac{a}{3}, 1\right]\right) = \mu(A \setminus R) = \frac{1}{3}. \tag{24}$$

Furthermore, we cannot be in Case 2 of the proof of Theorem 1 since the conclusion is then that $\mu(A) \leq 22/51 < 77/177$. Therefore we must be in Case 1 (and more precisely subcase $R_0 \neq \emptyset$) and several inequalities occurring in the course of the proof must actually be equalities. In particular, we must have

$$a = \frac{8}{177}, \quad r = \frac{2}{9} + \frac{a}{3} = \frac{14}{59}, \quad \text{and } b = \frac{2r}{9} + \frac{a}{3} = \frac{4}{59}.$$

Now, since $a > 0$, Lemma 5 shows that (24) implies $\varepsilon_1 = \varepsilon_2 = 0$, thus $\mu(A_1) = 1/3 = \mu(A \setminus R)$, therefore, up to a set of measure zero, we have

$$A = R \cup \left(\frac{2}{3}, 1\right). \tag{25}$$

Moreover, in the course of the proof of Theorem 1 we also applied Lemma 4 to $(1/r)\cdot R$, so, in the equality case, similar arguments as above yield, up to a set of measure zero,

$$R \cap [2r/9 + a/3, r] = (2r/3, r) = (28/177, 14/59).$$

What remains of A is, by definition, contained in $[a, b]$. It follows that up to a set of measure zero

$$A \subset \left(\frac{8}{177}, \frac{4}{59}\right) \cup \left(\frac{28}{177}, \frac{14}{59}\right) \cup \left(\frac{2}{3}, 1\right) = \mathcal{A}_0.$$

This implies the statement of the lemma since A and $\mathcal{A}_0$ have the same measure. □

Before coming to the proof of our inverse theorem, we recall a kind of prehistorical lemma in our context.

Lemma 8. *Let X and Y be two subsets of $\mathbb{R}$. Let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that $X \subset (\alpha, \beta)$, $\mu_*(X) = \beta - \alpha$, $Y \subset (\gamma, \delta)$, $\mu_*(Y) = \delta - \gamma$, then*

$$X + Y = (\alpha + \gamma, \beta + \delta).$$

Proof. Let $v \in (\alpha + \gamma, \beta + \delta)$. It can be written as $v = \phi + \chi$ with $\phi \in (\alpha, \beta)$ and $\chi \in (\gamma, \delta)$. Let

$$\theta = \frac{1}{2} \min(|\phi - \alpha|, |\phi - \beta|, |\chi - \gamma|, |\chi - \delta|).$$

It follows that $(\phi - \theta, \phi + \theta) \subset (\alpha, \beta)$ and $(\chi - \theta, \chi + \theta) \subset (\gamma, \delta)$. Since X and Y are of full measure, we must have $\mu_*(X \cap (\phi - \theta, \phi + \theta)) = \mu_*(Y \cap (\chi - \theta, \chi + \theta)) = 2\theta$. Moreover, the set $v - (\phi - \theta, \phi + \theta) = (\chi - \theta, \chi + \theta)$ and it follows that we must also have

$$\mu_*((v - X) \cap (\chi - \theta, \chi + \theta)) = \mu_*(Y \cap (\chi - \theta, \chi + \theta)) = 2\theta.$$

Consequently the two full-measure in $(\chi - \theta, \chi + \theta)$ sets $((v - X) \cap (\chi - \theta, \chi + \theta))$ and $Y \cap (\chi - \theta, \chi + \theta))$ must intersect which shows that there are a x in X and a y in Y such that $v - x = y$ or $v = x + y \in X + Y$. Hence the result, this being valid for any v. □

We start by a key-remark which will be essential in the proof of Theorem 2. Generalizing (6), we notice that if A is a 3-sum-free set, then we have

$$((3\cdot A)-A)\cap A=\emptyset. \tag{26}$$

Giving a proof is immediate.

We are now ready to conclude the proof of the Chung-Goldwasser conjecture and prove Theorem 2.

Proof of Theorem 2. Applying Lemma 7 gives that, under the hypothesis of the theorem, $\mu(A\Delta\mathcal{A}_0)=0$. Equivalently, A is of the form

$$A=U\cup V\cup A_1\cup Z$$

with

$$U\subset\left[\frac{8}{177},\frac{4}{59}\right],\quad V\subset\left[\frac{28}{177},\frac{14}{59}\right],\quad\text{and}\quad A_1\subset\left[\frac{2}{3},1\right],$$

these three sets being of maximal measure in their respective intervals; and $\mu(Z)=0$.

Having noticed that if a set is of full measure in a given interval then dilating it by a constant factor transforms it as a full measure set in the dilated interval, an easy computation, based on Lemma 8, shows that

$$(3\cdot V-V)\cup\left(\frac{1}{3}\cdot(A_1+A_1)\right)=\left(\frac{14}{59},\frac{98}{177}\right)\cup\left(\frac{4}{9},\frac{2}{3}\right)=\left(\frac{14}{59},\frac{2}{3}\right).$$

In the same way, we compute that

$$(3\cdot U)-U=\left(\frac{4}{59},\frac{28}{177}\right)$$

and

$$(3\cdot V)-A_1=\left(-\frac{31}{59},\frac{8}{177}\right).$$

By (26), the union of all these sets is intersection-free with A, therefore A is contained in its complementary set in $[0,1]$, namely

$$A\subset\left[\frac{8}{177},\frac{4}{59}\right]\cup\left[\frac{28}{177},\frac{14}{59}\right]\cup\left[\frac{2}{3},1\right].$$

It follows that $Z=\emptyset$.

Studying the different cases with the endpoints leads to the result. □

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E-mail address: plagne@math.polytechnique.fr
E-mail address: anne.de-roton@univ-lorraine.fr

CMLS, École polytechnique, 91128 Palaiseau Cedex, France

Université de Lorraine, Institut Elie Cartan de Lorraine, UMR 7502, Vandoeuvre-lès-Nancy, F-54506, France, CNRS, Institut Elie Cartan de Lorraine, UMR 7502, Vandoeuvre-lès-Nancy, F-54506, France.